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A Variational Approach for the Dynamics of Unsteady Point Vortices with Application to Impulsively Started Aerofoil

Ahmed Hussein* Haithem E. Taha † Saad Ragab ‡ Muhammad R. Hajj§

A Lagrangian formulation for the dynamics of unsteady point vortices is introduced and implemented. The proposed Lagrangian is related to previously constructed Lagrangian of point vortices via a gauge-symmetry in the case of vortices of constant strengths; i.e., they yield the exact same dynamics. However, a different dynamics is obtained in the case of unsteady point vortices. The resulting Euler-Lagrange equation derived from the principle of least action exactly matches the Brown-Michael evolution equation for unsteady point vortices, which was derived from a completely different point of view; based on conservation of linear momentum. The resulting dynamic model of time-varying vortices is applied to the problem of an impulsively started flat plate. The resulting lift coefficient using the dynamics of the proposed Lagrangian is compared to that using previously constructed Lagrangian and other models in literature.

I. Introduction

Reduced-order modeling of unsteady aerodynamics has been a topic of research interest since the early formulations of Prandtl [1] and Birnbaum [2]. These formulations were followed by the seminal works of Wagner [3] and Theodorsen [4]; the later efforts of Leishman [5, 6] and Peters [7, 8]; and in more recent papers by Ansari et al. [9, 10], Taha et al. [11] and Yan et al. [12] among others. Another hallmark in the history of unsteady aerodynamic modeling is the development of the vortex lattice method (UVLM) or the discrete vortex method (DVM) about fifty years ago because of its ability to account for deforming wakes associated with relatively large amplitude maneuvers, flexible wings, and arbitrary time-varying wing motions. In DVMs, a point vortex is released at each time step to satisfy the Kutta condition at the sharp edge it sheds from. Moreover, all of the shed vortices move with constant strengths that have been dictated at the shedding time by the Kutta condition. As such, Helmholtz conservation laws [13] dictate that the dynamics of these constant-strength point vortices will force them to convect with the fluid's local velocity, i.e. the Kirchhoff velocity, see Saffmann [14], pp. 10. Although DVMs were used to develop efficient numerical algorithms to solve for aerodynamic quantities associated with unsteady maneuvers, they require shedding point vortices at each time step, which increases the number of degrees of freedom considerably as the simulation time increases. As a remedy, it has been suggested to replace the continuous shedding of constant-strength point vortices with discontinuous/intermittent shedding of varying-strength point vortices, i.e. the strength of the most recent shed point vortex is adjusted each time step to satisfy the Kutta condition, instead of shedding a new vortex to achieve the same objective. Shedding is deactivated until the strength of the unsteady point vortex reaches an extremum [15, 16]. At that instant, a new point vortex is shed from the same edge and the previous vortex is convected downstream with the Kirchhoff velocity while keeping its strength constant.

A serious technical problem that is associated with varying-strength (unsteady), point vortices is the non-uniqueness of their dynamics. In particular, they cannot convect with the Kirchhoff velocity because this will lead to spurious forces on the branch cut between the point vortex and the shedding edge. In other words, the linear and angular momenta will not be conserved. Interestingly, this issue has been pointed in Prandtl's lectures [17], pp. 153, and first analyzed by Brown and Michael [18], and independently by Edwards [19], when developing an analytical potential flow model for the dynamics of leading-edge vortices on three-dimensional delta wings. In their formulation, Brown and Michael [18] proposed a general model for the dynamics of unsteady point vortices shed from sharp edges that removes the spurious force proportional to the time-derivative of the vortex strength. Later on, Cheng [20] and Rott [21] introduced the same concept for two dimensional flows with vortices of variable strengths. This model in conjunction with the above intermittent shedding criterion constituted the basis for the more recent efforts on reduced-order modeling of unsteady aerodynamics of maneuvering airfoils by Cortelezzi and Leonard [16] and Michelin and Smith [22]. Howe [23] derived a modified version of the Brown-Michael equation that accounts for the unbalanced surface forces.

*PhD student,Department of Biomedical and Engineering Mechanics, Virginia Tech., Blacksburg
†Assistant Professor, Department of Mechanical and Aerospace Development, University of California, Irvine
‡Professor, Department of Biomedical and Engineering Mechanics, Virginia Tech., Blacksburg
§Professor, Department of Biomedical and Engineering Mechanics, Virginia Tech., Blacksburg

He showed that although the trajectory of the unsteady point vortices slightly changes using the modified equation, the unbalanced surface forces resulted in reduction of the sound generated by the unsteady point vortices.

Recently, Tchieu and Leonard [24] proposed an alternative model to Brown-Michael's for the dynamics of unsteady point vortices. The model sets the convection velocity of the unsteady vortex such that the resulting impulse is the same as that of a surrogate constant-strength vortex moving with the Kirchhoff velocity. They applied it to the problem of impulsively started flat plate and showed that the model results in a lift behavior that is closer to Wagner's lift [3] than that of Brown-Michael's [18]. Wang and Eldredge [25] generalized the model proposed by Tchieu and Leonard [24] and named it the impulse matching model. They applied this model to the case of a pitching and perching of flat plate. Their results agreed well with experimental data [26] and high-fidelity vortex particle simulations [27]. Both the Brown-Michael model and the impulse matching model are intrinsically concerned only with conservation of the linear momentum, which precludes their extension to satisfy other important laws such as the conservation of angular momentum. We should note here that both models permit unbalance of the angular momentum [24, 25]).

Variational principles have been shown to be useful physical-based approaches for deriving governing equations of both solids and fluids [28, 29]. These equations are obtained by setting the first variation of the action, which is the time integral of a candidate Lagrangian function, to zero. Clebsh [28] and Hargreaves [30] derived the equations of motion for an inviscid, incompressible flow by defining the Lagrangian to be the integral of the fluid pressure. Later, Bateman [31] extended the principle to the case of compressible irrotational flow. Luke [32] showed that using variational principles, one is able to provide the boundary conditions by perturbing the limits of integration (Leibniz integral rule). Regarding the vortex motion, Bateman [31], followed by Serrin [33], showed that the equations of motion of vortex lines could be obtained from a variational approach with the ability to regularize the infinite velocity at the vortex center (Sec. 4 in Ref. [31]). These variational principles were also used to derive governing equations for the cases of fluid motion with distributed vorticity [34] or point vortices [35] with no boundaries, and for the case of a fluid-body interaction [36] that considered constant strength vortices only. Advances made in studying the Hamiltonian dynamics of point vortices [37, 35, 38] point to the potential of developing a variational principle governing the dynamics of unsteady point vortices interacting with a circular cylinder or a body conformal to it (e.g., airfoil), which is the objective of this work. Such a formulation will allow satisfaction of conservation laws via adding constraints to the variational problem. In addition, it will enable compact and efficient coupling with other variational principles governing rigid body and structural dynamics for coupled unsteady flight dynamics analysis and/or aeroelastic analysis. To date, there have been no developments for variational principles governing the dynamics of unsteady point vortices interacting with solid bodies enclosed by a non-zero total circulation.

The dynamics of constant-strength, point vortices in an inviscid fluid, which is governed by the Biot-Savart law, was derived by Chapman [37] from an action whose Lagrangian is the summation of two functions. The first function is a bilinear function in the vortex spatial coordinates and its velocity, and the second one is the Routh stream function. Recently, Shashikanth et al. [38] proved that the equations of motion for a cylinder moving in the presence of constant-strength vortices of zero sum (i.e., zero total circulation), known as Foppl problem [39, 40], have a Hamiltonian structure. Dritschel and Boatto [41] showed similar results for three dimensional differentiable surfaces conformal to a sphere.

In the present work, we present a new Lagrangian function for the dynamics of point vortices that is more general than Chapman's [37]. We examine the relation between the proposed Lagrangian and Chapman's Lagrangian for the cases of constant strength and time-varying point vortices. We compare the derived equations of motion to the governing equations derived by other approaches such as the Biot-Savart law for the case of constant strength vortices and the Brown-Michael model [18, 42], and the impulse matching model [24, 25] for the time-varying vortices. We apply the resulting dynamic model of time-varying vortices to the problem of an impulsively started flat plate.

II. Lagrangian Dynamics of Point Vortices

A. General Formulation

Considering the flow around a sharp-edged body (in the z-plane) and mapping it to the flow over a cylinder (in the ζ-plane) with an interrelating conformal mapping $z = z(\zeta)$, as shown in Fig. 1, the regularized local fluid velocity (Kirchhoff velocity) of the shed k^{th} vortex is given by [44, 45, 25]

$$\frac{dz_k}{dt} = w_k(z_k) = \frac{1}{[z'(\zeta_k)]^*} \lim_{\zeta \longrightarrow \zeta_k} \left[\frac{\partial F}{\partial \zeta} - \frac{\Gamma_k}{2\pi i} \frac{1}{\zeta - \zeta_k} - \frac{\Gamma_k}{4\pi i} \frac{z''(\zeta)}{z'(\zeta)} \right]^* \tag{1}$$

where F is the complex potential, Γ_k is the strength of the k^{th} vortex, and the asterisk refers to a complex conjugate. The last term on the right hand side, which involves the second derivative of the transformation, was first derived by

Routh then by Lin [44] and later by Clements [45].

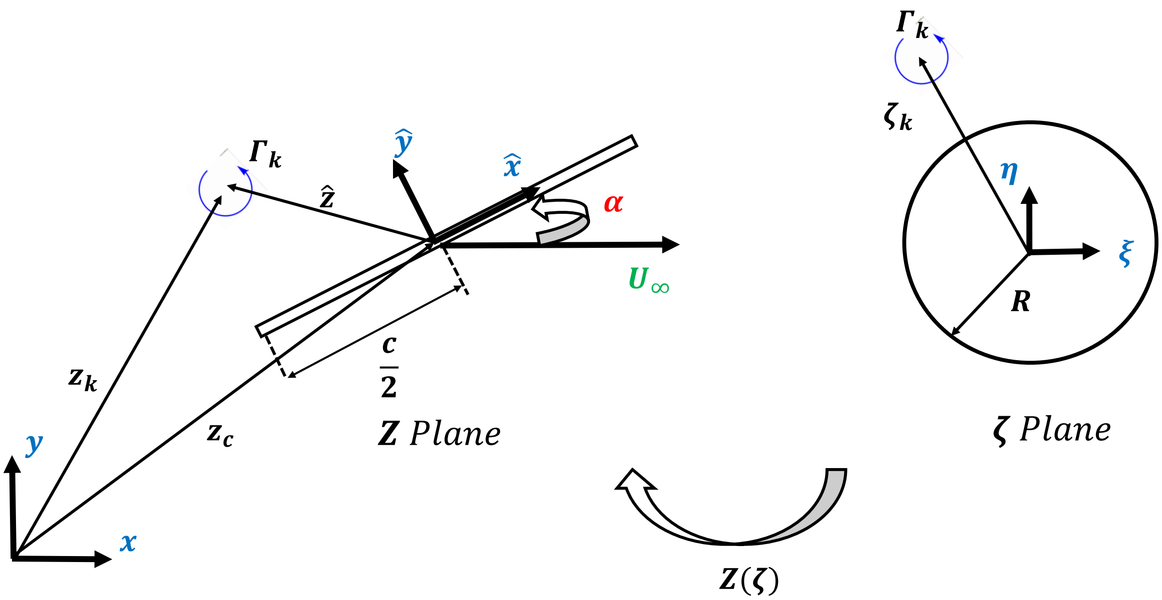


Figure 1: Conformal mapping between a sharp-edged body and a circular cylinder.

Lin [46] showed the existence of a Kirchhoff-Routh function W (Ref.[47] sec.13.48) that relates the velocity components of the k^{th} vortex to the derivatives of W, in a Hamiltonian form such that

$$\Gamma_k u_k = \frac{\partial W}{\partial y_k} \qquad \Gamma_k v_k = -\frac{\partial W}{\partial x_k} \tag{2}$$

The Kirchhoff-Routh function $\tilde{W}$ in the circle plane is related to the stream function ψ_0 by [44, 47, 48]

$$\tilde{W}(\xi_k, \eta_k) = \Gamma_k \psi_o(\xi_k, \eta_k) + \sum_{k,l,k\neq l} \frac{\Gamma_k \Gamma_l}{4\pi} \left[ln|\zeta_k - \zeta_l| - ln|\zeta_k - \zeta_l^I| \right] + \sum_k \frac{\Gamma_k^2}{4\pi} ln|\zeta_k - \zeta_k^{(I)}| \tag{3}$$

where ψ_o is the stream function of the body motion (i.e., $F = F_0 + \sum_{k=1}^n \Gamma_k$ and $F_0 = \phi_0 + i\psi_0$). Then the relation betwen the Kirchhoff-Routh function W in the flat plate plane and that in circle plane $\tilde{W}$ is given as [44] :

$$W = \tilde{W} + \sum_k \frac{\Gamma_k^2}{4\pi} ln|\frac{dz}{d\zeta}| \tag{4}$$

It is noteworthy that, as shown by Lin [46], the term ρW is a measure of the kinetic energy, where ρ is the density of the fluid. As such, the equations of motion can be determined from an energy minimization process. More details about the Hamiltonian structure of the motion of point vortices are provided by Aref [49].

B. Proposed Lagrangian of Point Vortices

We postulate a new Lagrangian function for the motion of point vortices in an infinite fluid in the z-plane in the most basic form as

$$L(z_k, z_k^*, \dot{z}_k, \dot{z}_k^*) = \frac{1}{i} \sum_{k=1}^n \Gamma_k z_k^* \dot{z}_k + W \tag{5}$$

where the first term is the bilinear function in variables z_k and $\dot{z}_k$, and the second term is the Routh stream function $W = -\frac{1}{2\pi}\sum_{k,l,k\neq l}\Gamma_k\Gamma_l \ln(z_k - z_l)(z_k - z_l)^*$. It has to be pointed out that the variable z_k and its conjugate z_k^* are treated as an independent variables. The bilinear nature of the first term ensures that the resulting equations of motion will involve only time derivatives of the first order. The same concept was introduced by Chapman whose Lagrangian is written as

$$\begin{aligned} L'(z_k, z_k^*, \dot{z}_k, \dot{z}_k^*) &= \frac{1}{2i}\sum_{k=1}^{n}\Gamma_k(z_k^*\dot{z}_k - z_k\dot{z}_k^*) - \frac{1}{2\pi}\sum_{k,l,k\neq l}\Gamma_k\Gamma_l ln(z_k - z_l)(z_k - z_l)^* \\ &= I_o + W \end{aligned} \tag{6}$$

where I_0 is one of the constants of motion associated with the motion of vortices of constant strengths in an infinite fluid. This Lagragian was then used in different contexts [50, 51].

It is interesting to note that the proposed Lagrangian L and Chapman's Lagrangian are related via a gauge symmetry for the case of constant-strength vortices. That is, we have

$$L' = L - \frac{1}{2i}\frac{d}{dt}\sum_{k=1}^{n}\Theta_k \tag{7}$$

where $\Theta_k = \Gamma_k z_k^* z_k$ is the angular momentum of the k^{th} vortex about the origin. Note that the gauge symmetry between any two Lagrangian functions such as L and L' implies that they add up to a total time derivative of some function, i.e., we have

$$L' = L + \frac{d}{dt}[F(q,t)]$$

where q are the generalized coordinates. As such, it is said that L and L' are related by a gauge symmetry or a gauge transformation and that both are gauge invariant [52, 53].

On the other hand, using Eq. (7), one may explain Chapman's Lagrangian L' as a constrained version of our proposed Lagrangian L to satisfy the constraint that the total angular momentum of the vortices about origin is conserved; i.e., $\frac{d}{dt}\sum_{k=1}^{n}\Theta_k = 0$.

C. Dynamics of a Constant Strength Point Vortices

To obtain the equations of motion for the case of vortices of constant strength, we define the action to be the integral of the Lagrangian

$$S = \int_{t_1}^{t_2} L(z_k, z_k^*, \dot{z}_k, \dot{z}_k^*)dt \tag{8}$$

Applying the principle of least action, i.e., setting the first variation of the action integral S to zero, the corresponding Euler-Lagrange equations are written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}_k}\right) - \frac{\partial L}{\partial z_k} = 0 \tag{9}$$

which yields the Biot-Savart law [54, 47, 55] that governs the motion of point vortices and is given by

$$\dot{z}_k^* = \frac{1}{2\pi i}\sum_{k,l,k\neq l}\frac{\Gamma_j}{z_k - z_l} \tag{10}$$

It should be noted that the same result can be obtained using Chapman's Lagrangian L' [37].

D. Dynamics of Unsteady Point Vortices Interacting with a Conformal Body

For a single point vortex of constant strength Γ, the Lagrangian proposed in Eq. (5) is written as

$$L(z, z^*, \dot{z}, \dot{z}^*) = \frac{1}{i}\Gamma z^*\dot{z} + W(z, z^*) \tag{11}$$

where $W(z, z^*)$ is the Kirchhoff-Routh function, which is a measure of the instantaneous energy in the flow [46] while accounting for the presence of the body. Allowing for a time-varying vortex strength (i.e. $\Gamma = \Gamma(t)$), a term that depends on the time rate of change of circulation (i.e. $\dot{\Gamma}$)) is added to ensure that the derivatives resulting from the bilinear function are coordinate-independent. As such, the Lagrangian is written as

$$L(z, z^*, \dot{z}, \dot{z}^*) = \frac{1}{i}\left(\Gamma z^* \dot{z} + \dot{\Gamma} z_0^* z\right) + W(z, z^*) \tag{12}$$

where z_0 is the coordinate of an arbitrary point on the body.

Now the Lagrangian of n point vortices of time-varying strengths is written as

$$L(z_k, z_k^*, \dot{z_k}, \dot{z_k}^*) = \frac{1}{i}\sum_{k=1}^{n}\left(\Gamma_k z_k^* \dot{z_k} + \dot{\Gamma} z_{0k}^* z_k\right) + W(z_k, z_k^*) \tag{13}$$

where z_{0k} is the coordinate of a reference point on the body, which is usually the coordinate of the edge from which the vortex is shed [18, 22, 24, 25].

Applying Euler-Lagrange equations (9) associated with minimizing the action integral based on this transformed Lagrangian (13), we obtain the dynamics of an unsteady point vortex as

$$\dot{z_k} + \frac{\dot{\Gamma}_k}{\Gamma_k}(z_k - z_{0k}) = \left(\frac{i}{\Gamma_k}\frac{\partial W}{\partial z_k}\right)^* \tag{14}$$

which reduces to the Biot-Savart law given by Eq. (10) if $\dot{\Gamma}$ is set to zero.

The right hand side of Eq. (14) can be represented in terms of the regularized local fluid velocity (Kirchoff velocity) $w^*(z_k)$, obtained from Eq. (1) as shown by [47], which is expressed as

$$\left(\frac{i}{\Gamma_k}\frac{\partial W}{\partial z_k}\right)^* = w^*(z_k) \tag{15}$$

Combining Eq. (14) and Eq. (15), we write

$$\dot{z_k} + \frac{\dot{\Gamma}_k}{\Gamma_k}(z_k - z_{0k}) = w^*(z_k) \tag{16}$$

which is exactly the same equation obtained by Brown and Michael [18] from a completely different approach that was based on a linear momentum argument.

It is interesting to note that while both the proposed Lagrangian L and Chapman's L' [37] yield the exact same dynamics, i.e. the Biot-Savart law for constant-strength vortices, they yield different dynamics for unsteady point vortices. Adding similar term to Chapman's Lagrangian L' to obtain a coordinate-independent expression for the vortex absolute velocity and minimizing the action integral based on this transformed Lagrangian, the resulting equation of motion is

$$\dot{z_k} + \frac{\dot{\Gamma}_k}{2\Gamma_k}(z_k - z_{0k}) = w^*(z_k) \tag{17}$$

which differs from that of Brown-Michael by the factor of one half that multiplies the $\dot{\Gamma}$-term.

Next, we apply the variational principle approach as defined above and evaluate the performance of both postulated and Chapman's [37] Lagrangians in predicting flow quantities. Particularly, we compare time histories of the circulation and lift coefficient to those obtained using the impulse matching model by Wang and Eldredge [25] and Wagner's function [3].

III. Impulsively Stared Flat Plate (The Starting Vortex Problem)

We consider a flat plate of semi-chord $c/2$ mapped from a circle of radius R, as shown in Fig. 1, according to the conformal mapping

$$z(\zeta) = z_c + g(\zeta)e^{i\alpha} \tag{18}$$

where the mapping function, g, is defined as

$$g(\zeta) = \zeta + \frac{R^2}{\zeta} \tag{19}$$

The derivative of z with respect to ζ is

$$\frac{dz}{d\zeta} = g'(\zeta)e^{i\alpha} \tag{20}$$

We also consider the case where the flat plate is moving with a constant speed U_∞ , inclined to the x-axis by an angle α. A vortex of strength Γ_v is shed from the trailing edge as shown in Fig. 1. For this flow, the complex potential in the circle plane is written as [47, 48, 22]

$$F(\zeta) = \phi(\zeta) + i\psi(\zeta) = V(\zeta - g(\zeta)) + \frac{R^2\bar{V}}{\zeta} + \frac{\Gamma_v}{2\pi i}\left[ln(\zeta - \zeta_v) - ln(\zeta - \zeta_v^{(I)})\right] \tag{21}$$

where ϕ is the velocity potential, ψ is the stream function, $V = -U_\infty e^{i\alpha}$ is the velocity of the flat plate in the plate-fixed frame, and $\zeta_v^I = R^2/\zeta_v^*$ denotes the position of the image vortex within the circle. The first term inside the brackets $(\zeta - g(\zeta))$ ensures that the complex potential will contain only ζ with negative power (see Sec. 9.63 [47] , Sec. 4.71 [54], Sec. 4 [48], Sec. 3.2 [22]).

A. Dynamics of the Starting Vortex

Taking the origin at the mid-chord point and assuming that the starting vortex shed from the trailing edge ($\hat{z}_{v0} = -c/2$), we write the evolution equation of the starting vortex according to the Lagrangian dynamics as

$$\begin{aligned} \dot{z}_v + \frac{\dot{\Gamma}_v}{\beta\Gamma}(z_v - z_{v0}) &= (\frac{i}{\Gamma}\frac{\partial W}{\partial z_v})^* \\ &= (\frac{i}{\Gamma}\frac{\partial W}{\partial \zeta_v}(\frac{dz}{d\zeta})^{-1}_{z_v})^* \\ &= w^*(z_v) \end{aligned} \tag{22}$$

where β is a factor used to differentiate between the equation obtained from the proposed Lagrangian L ($\beta = 1$) or Chapman's Lagrangian L' ($\beta = 2$). Also, we have

$$W(z_v) = \Gamma_v\psi_o + \frac{\Gamma_v^2}{4\pi}|\ln(\zeta_v - \zeta_v^{(I)})| + \frac{\Gamma_v^2}{4\pi}\ln|\frac{dz}{d\zeta}|_{z_v} \tag{23}$$

and

$$\psi_0 = Im\left(V(\zeta - g(\zeta)) + \frac{R^2V^*}{\zeta}\right) \tag{24}$$

Transforming Eq. (22) to the circle plane, the first term in the left hand side is written as

$$\dot{z}_v = U_\infty + g'(\zeta_v)e^{i\alpha}\dot{\zeta}_v \tag{25}$$

and the right hand side of Eq. (22) is re-written as

$$\begin{aligned} w^*(\zeta) =& \frac{e^{i\alpha}}{[g'(\zeta)]^*}\left[V(1 - g'(\zeta)) - \frac{R^2\bar{V}}{\zeta^2} - \frac{\Gamma_v}{2\pi i}\frac{1}{\zeta - \zeta_v^I} - \frac{\Gamma_v}{4\pi i}\frac{g''(\zeta)}{g'(\zeta)}\right]^* \\ =& \frac{e^{i\alpha}}{[g'(\zeta)]^*}\left[V - \frac{R^2\bar{V}}{\zeta^2} - \frac{\Gamma_v}{2\pi i}\frac{1}{\zeta - \zeta_v^I} - \frac{\Gamma_v}{4\pi i}\frac{g''(\zeta)}{g'(\zeta)}\right]^* - Ve^{-i\alpha} \end{aligned} \tag{26}$$

Recalling that $V = -U_\infty e^{i\alpha}$, we write

$$w^*(\zeta) = \frac{e^{i\alpha}}{[g'(\zeta)]^*}\left[V - \frac{R^2\bar{V}}{\zeta^2} - \frac{\Gamma_v}{2\pi i}\frac{1}{\zeta - \zeta_v^I} - \frac{\Gamma_v}{4\pi i}\frac{g''(\zeta)}{g'(\zeta)}\right]^* + U_\infty \tag{27}$$

The evolution equation is then re-written in terms of the circle-plane variables as

$$\dot{\zeta}_v + \frac{\dot{\Gamma}_v}{\beta\Gamma_v}\frac{(g(\zeta_v) - 2R)}{g'(\zeta_v)} = \frac{1}{g'(\zeta_v)[g'(\zeta_v)]^*}\left[V - \frac{R^2\bar{V}}{\zeta_v^2} - \frac{\Gamma}{2\pi i}\frac{1}{\zeta_v - \zeta_v^I} - \frac{\Gamma_v}{4\pi i}\frac{g''(\zeta_v)}{g'(\zeta_v)}\right]^* \tag{28}$$

A more general form of Eq. (28), for $\beta = 1$, for a flat plate moving and rotating in space can be found in the work of Michelin and Smith [22].

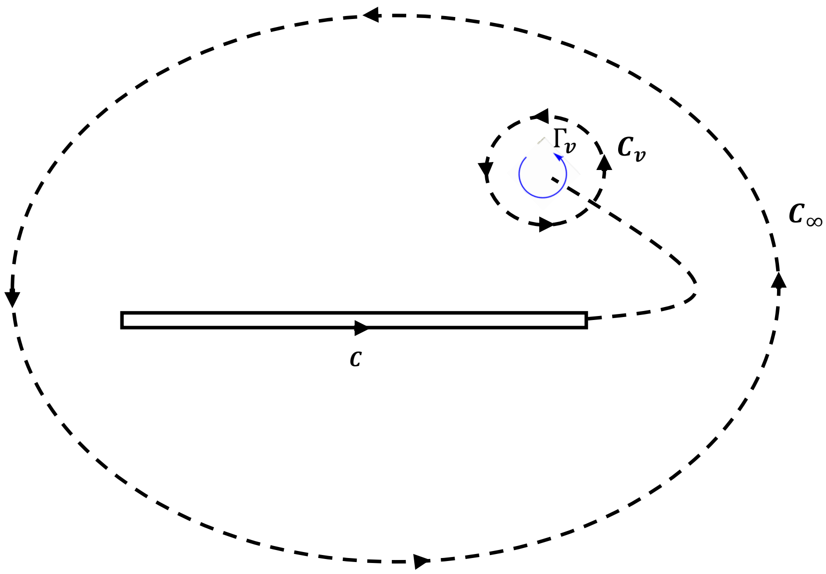


Figure 2: The contour used to evaluate the integral on the solid body. The flat plate is exaggerated to show the direction of the contour

B. Aerodynamic Forces

The force on the flat plate is obtained using the force formula derived by Sedov [56], in terms of the complex variable z, as

$$F_x + iF_y = -i\rho z_o \frac{d\Gamma_v}{dt} + \overline{\frac{i\rho}{2}\int_C [w(z)]^2 dz} + \frac{d}{dt}\left[i\rho \int_C zw(z)dz\right] \tag{29}$$

Using Cauchy's theorem [57], the integration can be changed from an integration over the solid body C to an integration over the infinite domain C_∞ that excludes the integration over an infinitesimally small contour C_v around the vortex (see sec.3.4.1 in Ref. [22]) as shown in Fig. 2. Upon evaluating the integration, the force in terms of the complex variable ζ becomes

$$F_x + iF_y = i\rho e^{i\alpha}\left[2i\pi R^2 Im(V) + \frac{d}{dt}\left(\Gamma_v(\zeta_v - \frac{R^2}{\bar{\zeta_v}})\right)\right] \tag{30}$$

The vortex strength is calculated by satisfying the Kutta condition at each time instant. The Kutta condition is implemented by requiring that the tangential velocity at the trailing edge in the circle plane vanishes; i.e., the terms inside the brackets in Eq. (28) are set to zero at the trailing edge to cancel the singularity due to $1 - R^2/\zeta_{v0}^2 = 0$. This will ensure a finite velocity at the trailing edge. As such, we write

$$V - \frac{R^2\bar{V}}{\zeta_0^2} + \frac{\Gamma_v}{2\pi i}(\frac{1}{\zeta_0 - \zeta_v} - \frac{1}{\zeta_0 - \zeta_v^I}) = 0 \tag{31}$$

Equation (31) is then re-written as

$$2iIm(V) + \frac{\Gamma_v}{2\pi i}\left(\frac{-R^2 + (\eta_v + \xi_v)^2}{2(\eta_v^2 + (\xi_v - 2)^2)}\right) = 0 \tag{32}$$

where ξ_v and η_v are the real and imaginary parts of ζ_v.

By simple manipulation, Eq. (32) is re-written in a simple form as [22, 25]

$$2Im(V) + \frac{\Gamma_v}{2\pi}Re\left(\frac{\zeta_{v0} + R}{\zeta_{v0} - R}\right) = 0 \tag{33}$$

For the force calculations using the impulse of the starting vortex, the reader is referred to section 3.10 of Ref. [14].

IV. Numerical Results

One type of airfoil motion is considered in this section: an impulsively started motion in which the airfoil is suddenly accelerated to velocity U_∞. For integrating the equations of motion, we used the Matlab solver **ode15s** with a fixed time step of $\Delta t = 10^{-5} c/U_\infty$. This solver shows a better performance than others because of the stiff nature of the evolution equation. For the first time step, instead of integrating the equations of motion analytically along with the Kutta condition as in Refs [22, 58], we used an appropriate initial condition for the position of the vortex , i.e. $x(0) = c/2 + \epsilon$, where $\epsilon \approx 10^{-4}c$.

A. Impulsively Started Flat Plate

First, similar to the classical unsteady thin airfoil theory (e.g., Wagner [3], Theodorsen [4], and Von Karman and Sears [59]), we assume that the starting vortex moves along the x-axis and the local fluid velocity is U_∞ (i.e., $w(z_v) = U_\infty$). As such, the evolution equation (22) in the z planes is written as

$$\dot{x}_v + \frac{\dot{\Gamma}_v}{\beta \Gamma_v}(x_v - x_{v0}) = U_\infty \tag{34}$$

The evolution equation of the impulse matching model [24, 25] can also be simplified to

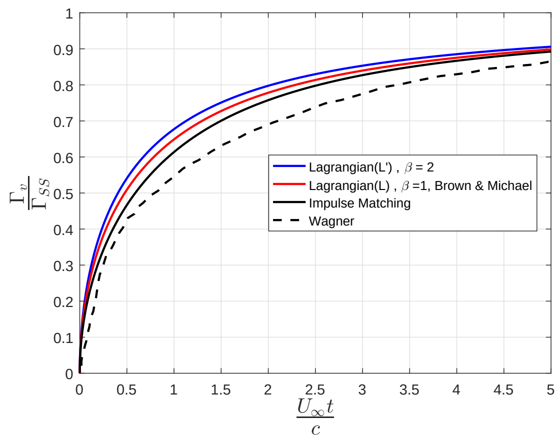
$$\dot{x}_v + \frac{\dot{\Gamma}_v}{\Gamma_v}\frac{(x_v^2 - x_{v0}^2)}{x_v} = U_\infty \tag{35}$$

Figure 3 shows the time variations of the normalized vortex strength Γ, the lift coefficient C_L, and the time-variation of the normalized vortex location x for the case of $\alpha = 5^o$. Plots from simulations based on (i) the proposed Lagrangian dynamics ($\beta = 1$ Brown-Michael), (ii) Chapman's Lagrangian ($\beta = 2$), (iii) the impulse matching model of Wang and Eldredge [25], and (iv) Wagner's [3] step response function are presented for the sake of comparison. The plots show that all models agree qualitatively with Wagner's exact potential flow solution. Note that in the three models, the infinite sheet of wake vorticity is approximated by a single vortex. As expected, the correction to the Kirchhoff velocity (taken as U_∞ here) in the case of $\beta = 2$ is half of that in the case of $\beta = 1$ yields slightly higher (spurious) lift.

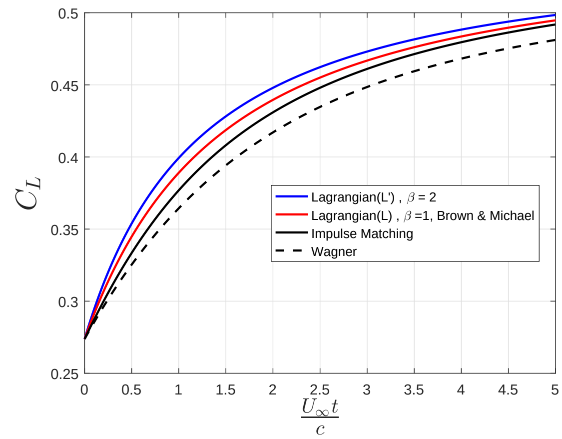
Next, we consider increasing the angle of attack to $\alpha = 10°$ to relax the flat wake assumption. Thus allowing the vortex to move in the plane, i.e. with two degrees of freedom. Figure 4 shows the resulting time variations of the normalized circulation Γ, lift coefficient C_L, vortex position along the x-axis, and the slope of the vortex trajectory θ as a function of x. The singular value of the lift at $t = 0$, which corresponds to the added mass effect, is removed to highlight the difference between results from different models. Again, the results based on L' ($\beta = 2$) predict a larger vortex strength (airfoil circulation) and a slightly higher lift, than those predicted by the two other models. Figure [4d] shows that the slope of the starting vortex asymptotically approaches a line parallel to the incident free stream (i.e. $\theta \approx \alpha = 10^o$). As shown, the proposed Lagrangian (Brown-Michael model) yield lift and circulation values that do no match Wagner's function. In addition, the impulse matching results in a slower downstream convection. Consequently the development of circulation takes place at a slower rate with an overall effect of reduced lift coefficient that matches Wagner's function. We note, however, that the Wagner's response should not be considered as a reference for comparison in this case because of the flat-wake and shedding by U_∞ assumptions that may not be appropriate for the relatively large angle of attack. This can be seen from the high-fidelity results in Fig. 4b as the lift starts to disagree with Wagner's function after non-dimensional time of $U_\infty t/c > 1.2$. The high-fidelity results was produced in Ref.[25] using the viscous vortex particle method developed by Eldredge [27].

V. Conclusions

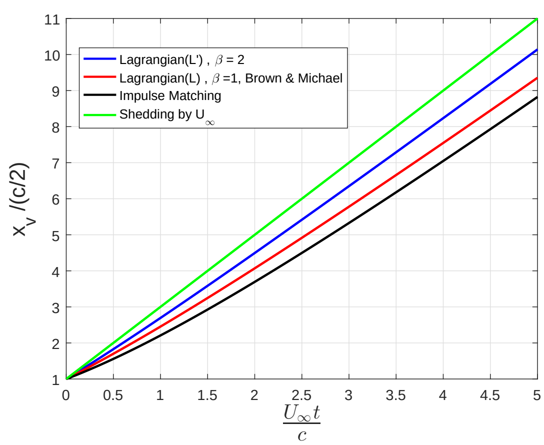
We investigated the potential of implementing variational principles to derive governing equations for the interaction of unsteady point vortices with a solid boundary. To do so, we postulated a new Lagrangian function for the dynamics of point vortices that is more general than Chapman's. We showed that this function is related to Chapman's Lagrangian via a gauge symmetry for the case of constant-strength vortices. In other words, both Lagrangian functions result in the same governing equation, i.e. the Biot-Savart law is directly recovered from the Euler-Lagrange equations corresponding to minimization of the action integral with these two Lagrangians. We also found that, unlike Chapman's Lagrangian, the principle of least action based on the proposed Lagrangian results exactly in the Brown-Michael model for the dynamics of unsteady point vortices. We implemented the resulting dynamic model of time-varying vortices to the problem of an impulsively started flat plate.



a) Time variation of the normalized vortex strength Γ_v. The circulation is normalized with the steady-state value Γ_{SS}.

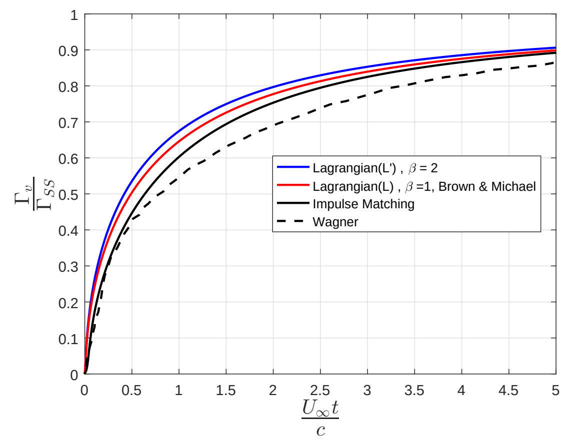


b) Time variation of the lift coefficient C_L

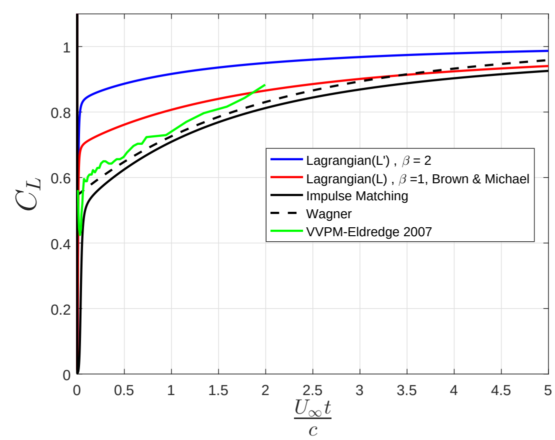


c) Time variation of the normalized vortex position x_v. The position is normalized using the semi-chord of the airfoil.

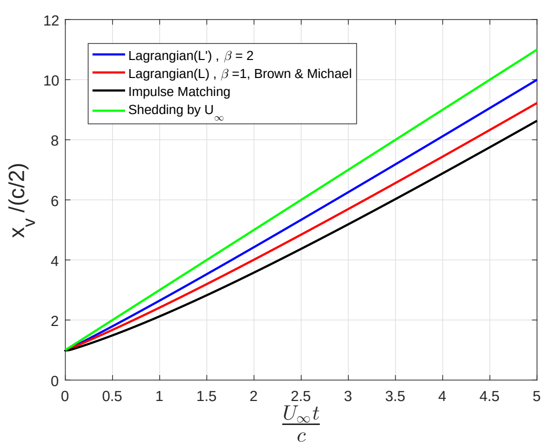
Figure 3: Time variations of (a) the normalized circulation, (b) lift coefficient and (c) normalized position of the starting vortex for $\alpha = 5^o$ and the vortex is assumed to move only in the x direction. The time is normalized using the airfoil speed U_∞ and chord c.



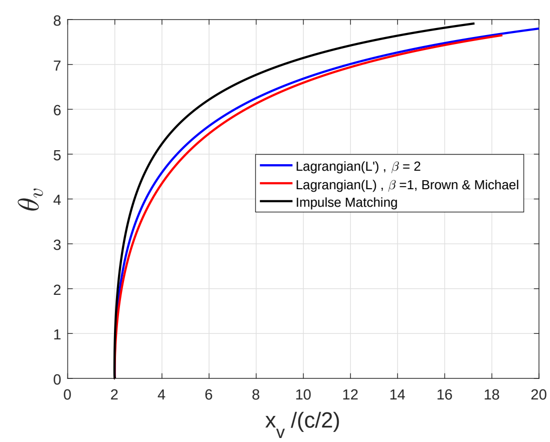
a) Time variation of the normalized vortex strength Γ_v. The circulation is normalized with the steady-state value Γ_{SS}



b) Time variation of the lift coefficient C_L



c) Vortex position x_v versus non-dimensional time



d) Slope of the vortex trajectory θ_v versus vortex position x_v

Figure 4: Time variations of (a) the normalized circulation, (b) lift coefficient, (c) normalized position of the starting vortex, and the slope of the vortex trajectory for $\alpha = 10^o$ and the vortex is allowed to move freely in the plane of the airfoil. The time is normalized using the airfoil speed U_∞ and chord c.

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